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Heterotic Cosets[†]

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Abstract

A description is given of how to construct $(0, 2)$ supersymmetric conformal field theories as coset models. These models may be used as non-trivial backgrounds for Heterotic String Theory. They are realised as a combination of an anomalously gauged Wess–Zumino–Witten model, right-moving supersymmetric fermions, and left-moving current algebra fermions. Requiring the sum of the gauge anomalies from the bosonic and fermionic sectors to cancel yields the final model. Applications discussed include exact models of extremal four-dimensional charged black holes and Taub–NUT solutions of string theory. These coset models may also be used to construct important families of $(0, 2)$ supersymmetric Heterotic String compactifications. The Kazama–Suzuki models are the left–right symmetric special case of these models.

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1. Introduction and Motivation

My aim here is to show how to construct non-trivial conformal field theories with $(0, 2)$ supersymmetry. The motivation is clear: It is well known that in order to obtain the desired $N = 1$ spacetime supersymmetry in heterotic string theory, the minimum requirement is world sheet $N = 2$ supersymmetry. Well, we have heard in the School about the highly studied $(2, 2)$ conformal field theories and many of the fascinating facts about their moduli spaces (e.g. Mirror Symmetry). However, these models are in a sense over-specialised examples of the generic $(0, 2)$ supersymmetric conformal field theories which heterotic string theory demands. In this sense, the task of studying the moduli space of heterotic string vacua has only just begun. A search for many $(0, 2)$ models and understanding of their moduli has to begin in earnest. This talk will describe the construction of isolated points in this moduli space. I first describe the general case and end with some examples and a brief discussion.

2. $(2,2)$ cosets: Kazama–Suzuki Models

Coset models were first invented by Bardakci and Halpern[1] and later generalised by Goddard, Kent and Olive[2] as algebraic realisations of new conformal systems, ‘ G/H ’ based upon affine Lie algebras (a special case of Kac–Moody algebras[3][4]) for a group G and a subgroup H . The $N = 1$ supersymmetric extension was worked out soon after and is based upon analogous constructions using affine Lie superalgebras[5]. (For a review see ref.[6].)

When the space G/H is Kähler, it was shown by Kazama and Suzuki[7] using an algebraic construction that $N = 1$ is promoted to an $N = 2$ supersymmetry. This realises a large family of $(2, 2)$ models, of which the $N = 2$ minimal models (used for example by Gepner in his construction of non-trivial heterotic string vacua) are the simplest case (they are realised as $SU(2)/U(1)$). The most straightforward examples are the ‘Hermitian Symmetric Spaces’.

For many reasons it is advantageous to have a Lagrangian definition of a conformal field theory which realises the algebraic structures as a field theory. It is very often

a powerful supplement to the algebraic description. The *Gauged Wess–Zumino–Witten Model* is the appropriate device to use.

3. (2,2) Cosets as Gauged Wess–Zumino–Witten Models

An action for a conformal field theory with all of the algebraic structures of the Kazama–Suzuki models is:

$$\begin{aligned}
I^{(2,2)} = & I_{WZW}(g) + I(g, A) + I_F(\Psi_L, \Psi_R, A) = \\
& - \frac{k}{4\pi} \int_{\Sigma} d^2z \operatorname{Tr}[g^{-1} \partial_z g \cdot g^{-1} \partial_{\bar{z}} g] \\
& - \frac{i}{12\pi} \int_B d^3\sigma \epsilon^{ijk} \operatorname{Tr}[g^{-1} \partial_i g \cdot g^{-1} \partial_j g \cdot g^{-1} \partial_k g] \\
& + \frac{k}{2\pi} \int_{\Sigma} d^2z \operatorname{Tr}[A_z g^{-1} \partial_z g - A_{\bar{z}} \partial_{\bar{z}} g g^{-1} + A_{\bar{z}} g^{-1} A_z g - A_z A_{\bar{z}}] \\
& + \frac{i}{4\pi} \int_{\Sigma} d^2z \operatorname{Tr}[\Psi_+ \mathcal{D}_{\bar{z}} \Psi_+ + \Psi_- \mathcal{D}_z \Psi_-]
\end{aligned} \tag{3.1}$$

where $\Sigma = \partial B$ and

$$\begin{aligned}
g & \in G; \quad A^a \in \operatorname{Lie} H; \\
\Psi_{\pm} & \in \operatorname{Lie} G - \operatorname{Lie} H, \quad \mathcal{D}_a \equiv \partial_a + [A_a, \]
\end{aligned}$$

and we have gauged the group invariance

$$\begin{aligned}
g & \rightarrow h g h^{-1}, \\
\text{where } h & \in H.
\end{aligned}$$

This action has an $N = 1$ supersymmetry:

$$\begin{aligned}
\delta g & = i\epsilon_- g \Psi_+ + i\epsilon_+ \Psi_- g \\
\delta \Psi_+ & = \epsilon_- (1 - \Pi_0) \cdot (g^{-1} \mathcal{D}_z g - i\Psi_+ \Psi_+) \\
\delta \Psi_- & = \epsilon_+ (1 - \Pi_0) \cdot (\mathcal{D}_{\bar{z}} g g^{-1} + i\Psi_- \Psi_-)
\end{aligned} \tag{3.2}$$

where Π_0 is the orthogonal projection of $\operatorname{Lie} G$ onto $\operatorname{Lie} H$.

Now, just as in the algebraic construction of Kazama and Suzuki, an $N = 2$ supersymmetry arises from this $N = 1$ when the space G/H is Kähler. I will not

dwelling on this further here, save to note that this action was first studied in this context by Witten[8] and Nakatsu[9]. Witten used this action (after twisting) to do explicit calculations in certain topological field theories. The explicit $N = 2$ transformations are written down in ref.[10] for example and there Henningson uses the models to study important properties of the Kazama–Suzuki models which are more easily accessible via field theoretic methods. This includes a demonstration of mirror symmetry for the Kazama–Suzuki models and a calculation of the elliptic genus for the $N = 2$ minimal models.

Note by the way that the bosonic and fermionic sectors in (3.1) are consistent models. In particular, the bosonic sector of the gauged Wess–Zumino–Witten model is of course a consistent model realising the bosonic cosets[11] and the action for the chiral fermions, when written in this ‘coset’ basis, is just a simple minimal coupling to the gauge fields[12]. The chiral anomalies which potentially arise from this coupling exactly cancel due to the identical nature of the left and right fermion couplings. The anomalies contribute with opposite sign but equal magnitude.

4. (0,2) Cosets: Potential Problems and a Solution

(1) To get a $(0, 2)$ conformal field theory, we need to remove the left $N = 2$. Simply deleting or changing the couplings of the left moving fermions to the gauge fields would certainly do this for us, without spoiling the right-moving $N = 2$. The only problem is that this procedure is bound to produce anomalies. The right-movers’ chiral anomaly will either have nothing to cancel against (if we deleted the left-movers), or will not completely cancel (if we changed the couplings of the left-movers to spoil the third symmetry in (3.2)).

(2) For many other reasons (as will be illustrated later), it would also be nice to gauge other symmetries of the WZW model. To get a consistent model, one has to gauge a restricted class of subgroups of the full $G_L \times G_R$ symmetry which exists for the basic WZW. These are called ‘anomaly-free’ subgroups, mainly because one of the first uses of this type of model (in higher dimensional gauge theories) was to study the structure of anomalies[13] by deliberately studying anomalous subgroups,

and then letting the Wess–Zumino term produce *classically* the familiar quantum gauge anomalies. Since Witten’s paper on the use of the Wess–Zumino term to define a conformally invariant sigma model in two dimensions—the Wess–Zumino–Witten model—most of the efforts involving them in 2D, including their gauged versions, have made sure that there are no anomalies. This is simply because the model would not correctly reproduce the coset algebra—it would not be conformally invariant, in general.

Given the language I just used to describe the problems we would like to solve, it is clear that a solution presents itself in the form of *cancelling the anomalies against one another*. If we arrange things correctly, this will work. In the next section I describe just how to do this.

5. Anomalies

There are anomalies arising from three sectors now. The classical anomaly from the WZW and the chiral anomalies at one-loop from each chirality of fermion. I will discuss each in turn.

The WZW anomalies.

In general gauging the following symmetry of the WZW model

$$g \rightarrow h_1 g h_2^{-1}$$

$$\text{for } (h_1, h_2) \in (H_L, H_R) \subset (G_L, G_R)$$

is anomalous. This simply means that one cannot write down an extension of the WZW model which promotes this symmetry to a local invariance: There will always be terms which spoil gauge invariance. (This is because of the Wess–Zumino term; the metric term may be simply minimally coupled.)

Knowing that we will get an anomaly, let us choose to write *some* gauge extension such that under gauge transformations the ‘anomalous’ piece does not depend upon

the group element g . This results in the anomalous piece taking the form of the standard 2D chiral anomaly. The *unique* action is[14]:

$$\begin{aligned}
I_{GWZW}^{G_k}(g, A) = & -\frac{k}{4\pi} \int_{\Sigma} d^2z \text{Tr}[g^{-1}\mathcal{D}_z g \cdot g^{-1}\mathcal{D}_{\bar{z}}g] \\
& -\frac{i}{12\pi} \int_B d^3\sigma \epsilon^{ijk} \text{Tr}[g^{-1}\partial_i g \cdot g^{-1}\partial_j g \cdot g^{-1}\partial_k g] \\
& -\frac{k}{4\pi} \int_{\Sigma} A^a \wedge \text{Tr}[t_{a,L} \cdot dg g^{-1} + t_{a,R} g^{-1} dg] \\
& -\frac{k}{8\pi} \int_{\Sigma} A^a \wedge A^b \text{Tr}[t_{a,R} g^{-1} t_{b,L} g - t_{b,R} g^{-1} t_{a,L} g].
\end{aligned} \tag{5.1}$$

Under the infinitesimal variation

$$\begin{aligned}
g & \rightarrow g + \sum_a \epsilon_a (t_{a,L} g - g t_{a,R}) \\
A_z^a & \rightarrow A_z^a + \mathcal{D}_z \epsilon^a \\
A_{\bar{z}}^a & \rightarrow A_{\bar{z}}^a + \mathcal{D}_{\bar{z}} \epsilon^a,
\end{aligned}$$

the variation is

$$\begin{aligned}
\delta I(g, A) = & \frac{k}{4\pi} \text{Tr}[t_{a,R} \cdot t_{b,R} - t_{a,L} \cdot t_{b,L}] \int_{\Sigma} d^2z \epsilon^{(a)} F_{z\bar{z}}^{(b)} \\
& \text{where } t_{a,L(R)} \in \text{Lie} H_{L(R)}.
\end{aligned} \tag{5.2}$$

Notice in particular that for the popular diagonal gaugings of WZW models this variation is zero and the action reduces to the familiar one.

The right movers

As mentioned before, it is sufficient to minimally couple the coset fermions to the gauge fields:

$$\begin{aligned}
I_F^R(\Psi_R, A) = & \frac{k}{4\pi} \int_{\Sigma} i \text{Tr}[\Psi_R \mathcal{D}_{\bar{z}} \Psi_R] \\
& \text{where } \mathcal{D}_{\bar{z}} \Psi_R = \partial_{\bar{z}} \Psi_R + \sum_a A_{\bar{z}}^a [t_{a,R}, \Psi_R], \quad \Psi_R \in \text{Lie} G - \text{Lie} H.
\end{aligned} \tag{5.3}$$

There are $D = \dim G - \dim H$ fermions ψ_R^i in Ψ_R , all coupled with charges derived from the generators $t_{a,R}$. The chiral anomalies appear at one loop and are:

$$\frac{D}{4\pi} \text{Tr}[t_{a,R} \cdot t_{b,R}] \int_{\Sigma} d^2z \epsilon^{(a)} F_{z\bar{z}}^{(b)}. \tag{5.4}$$

(Note here the absence of k , which plays the role of $1/\hbar$. This really is a one loop effect.)

The left movers

Let us couple into the model some left movers. Let us add $D = \dim G - \dim H$ of them (a good choice, as we will see later) with arbitrary couplings. To be precise, arrange them into a fundamental vector $\Lambda_L = \{\lambda_L^i\}$ of the group $SO(D)_L$ which acts on them as a global symmetry, and minimally couple them to the H_L subgroup with generators $Q_{a,L}$ in this fundamental representation:

$$I_F^L(\lambda_L^i, A) = \frac{k}{4\pi} \int_{\Sigma} i \Lambda_L^T (\partial_z + \sum_a A_z^a Q_{a,L}) \Lambda_L. \quad (5.5)$$

(Here $\tilde{\text{Tr}}$ is the trace in the fundamental representation of $SO(D)$.)

Their chiral anomalies appear at one loop and are:

$$-\frac{1}{4\pi} \tilde{\text{Tr}}[Q_{a,L} \cdot Q_{b,L}] \int_{\Sigma} d^2 z \epsilon^{(a)} F_{z\bar{z}}^{(b)}. \quad (5.6)$$

(Note again the absence of k . Also note the minus sign relative to (5.4), due to the opposite chirality.)

So if we add together the three actions (5.1), (5.3) and (5.5), we get a gauge invariant model if we ensure that all of the anomalies (classical and quantum) cancel:

$$k \text{Tr}[t_{a,R} \cdot t_{b,R} - t_{a,L} \cdot t_{b,L}] + \text{Tr}[t_{a,R} \cdot t_{b,R}] - \tilde{\text{Tr}}[Q_a \cdot Q_b] = 0. \quad (5.7)$$

Our model has $(0, 2)$ supersymmetry as advertised (because we have not touched the right-moving sector), and is conformally invariant.

Well, our model is gauge invariant when we take into account the one-loop effects, but we still have not written a *classically* gauge invariant action. This means that we cannot truly carry out procedures like path-integral quantisation, etc. We have not quite achieved our goal of a Lagrangian realisation of a $(0, 2)$ conformal field theory.

The answer is to *bosonize* the fermions. The bosonic action equivalent to $I_R^F + I_L^F$ is *classically* anomalous. It is a theory of $D/2$ real bosons with the same anomalies as above.

6. Bosonisation

I will give specific examples later, where I have worked out the bosonisation by hand in some abelian cases. After a little thought, however, it is clear once one realises that a classically anomalous bosonic theory equivalent to an anomalous fermionic theory is to be found, it might be that the bosonic theory is something like another anomalously gauged WZW.

Note that before gauging there are D free fermions on the left and right. They therefore carry a global $SO(D)_L \times SO(D)_R$ symmetry. Witten showed in ref.[15] that this system of free fermions is equivalent to a Wess–Zumino–Witten model based on $SO(D)$ at level 1. Considering what we saw about WZW anomalies in earlier section it is clear that the classically anomalous bosonic theory equivalent to the fermionic theory is just this $SO(D)$ WZW gauged anomalously with different embeddings of H in $SO(D)$ on the left and on the right:

$$\begin{aligned} \tilde{g} &\rightarrow \tilde{h}_1 \tilde{g} \tilde{h}_2 \\ \text{for } \tilde{g} &\in SO(D) \text{ and} \\ (\tilde{h}_1, \tilde{h}_2) &\in (H_L, H_R) \subset (SO(D)_L, SO(D)_R) \end{aligned}$$

Let the (H_L, H_R) be generated by $(Q_{a,L}, Q_{a,R})$. Choose the $Q_{a,R}$ such that when acting on the ψ_R^i 's in the fundamental representation of $SO(D)$ they are equivalent to the $t_{a,R}$ acting on the ψ_R^i in the coset fermion $\Psi_R \in \text{Lie}G - \text{Lie}H$. This will ensure that the right moving fermions are correctly coupled and preserve the (now hidden) $N = 2$ on the right.

Then the bosonic action equivalent to the interacting fermions is just an action of the form (5.1) (with level 1), which yields the classical anomalies:

$$\frac{1}{4\pi} \tilde{\text{Tr}}[Q_{a,R} \cdot Q_{b,R} - Q_{a,L} \cdot Q_{b,L}] \int_{\Sigma} d^2z \, \epsilon^{(b)} F_{z\bar{z}}^{(a)}.$$

So cancelling this against the anomaly of the G/H bosonic model (and recalling from the above paragraph that $\tilde{\text{Tr}}[Q_{a,R} \cdot Q_{b,R}] = D\text{Tr}[t_{a,R} \cdot t_{b,R}]$), we recover (5.7) as the condition for a consistent model.

7. (0,2) Cosets as Gauged Wess–Zumino–Witten Models

So finally we can write a classically gauge invariant analogue of (3.1) which realises a $(0, 2)$ conformal field theory as a gauge invariant action written as the sum of two gauged Wess–Zumino–Witten models which are separately anomalous:

$$I^{(0,2)} = I_{GWZW}^{G_k}(g, A) + I_{GWZW}^{SO(D)_1}(\tilde{g}, A), \quad (7.1)$$

where $D = \dim G - \dim H$.

The heterotic coset is realised as: $[G_k \times SO(D)_1] / H$ with the gauged symmetry:

$$\begin{aligned} g &\rightarrow h_2 g h_1^{-1} \\ \tilde{g} &\rightarrow \tilde{h}_2 \tilde{g} \tilde{h}_1^{-1} \end{aligned}$$

$$\text{subject to } k\text{Tr}[t_{a,R} \cdot t_{b,R} - t_{a,L} \cdot t_{b,L}] + D\text{Tr}[t_{a,R} \cdot t_{b,R}] - \tilde{\text{Tr}}[Q_{a,L} \cdot Q_{b,L}] = 0.$$

Note that h_1 and \tilde{h}_1 are chosen so as to recover right–moving supersymmetry in the fermion picture.

Note that in (7.1) the gauge extensions to each WZW (written using (5.1)) are generally not gauge invariant, but together they are because of the anomaly equation above. In the special case of $h_2 = h_1$ and $\tilde{h}_2 = \tilde{h}_1$, they are each separately gauge invariant extensions, the anomaly equation is trivially satisfied, and we recover the $(2, 2)$ case. *In this sense, the $(2, 2)$ models can now be seen as a special case of a more general class of $(0, 2)$ models.*

8. Some examples.

I originally used these ideas to study some particular cases[16]. The prototype model for this construction is the ‘monopole theory’ of Giddings, Polchinski and Strominger[17] (GPS). It is a conformal field theory of a heterotic string in a Dirac

monopole background of charge Q on a two-sphere of radius of order Q . GPS described it as an asymmetric orbifold of $SU(2)$. Here, described as a heterotic coset, it is based upon an $SU(2)$ WZW with the $U(1)$ subgroup of the right $SU(2)$ gauged. Adding supersymmetric right movers and left movers of charge Q gives an anomaly equation $k = 2(Q^2 - 1)$. Bosonising the fermions it is possible to correctly determine the quadratic terms in the gauge fields which turns out to only depend upon Q . After integrating out the gauge fields, and correctly re-fermionising the action, the heterotic sigma model describing the above system is recovered. This is described in detail in ref.[16]. As pointed out by GPS, the tensor product of this model with a supersymmetric $SL(2, \mathbb{R})/U(1)$ 2D black hole coset[18] yields a 4D solution which is the extremal limit of the magnetically charged dilaton black hole of Gibbons, Maeda and Garfinkle, Horowitz and Strominger[19].

Notice that in the construction I described for the monopole theory, one cannot have a charge $Q = 0$ solution, as then the anomaly equation would not be satisfied. After a little thought, it is clear that there is a quick way out of this problem: simply gauge $g \rightarrow hg$ instead and keep everything else the same. Then the sign of the WZW anomaly changes and the condition $k = 2(1 - Q^2)$ should now be satisfied. Now it is possible to get a $Q = 0$ solution. (In constructing their neutral solution in their paper, Giddings, Polchinski and Strominger arrive at this simple modification in an equivalent way. This is indeed the same solution). Now naively, the interpretation of the model would be as a heterotic string on a neutral two-sphere background. However, it is easy to see that this is wrong. The problem of incorrectly identifying the two-sphere as the background manifold for small Q has its roots in the fact that the final form of the metric for the model is obtained by integrating out the constraining 2D gauge fields, a process which is well defined only for large Q . which is equivalent to small α' , or large k . Here, the neutral solution has $k = 2$, and no sensible metric interpretation may be made of the target space via perturbation theory, as all length scales (in units of α') contribute equally to the β -function equations.

The most obvious application of this construction at the time was to find more general 4D solutions which were dyons (both magnetic and electric charge). Applying this construction to general gaugings of $SL(2, \mathbb{R})$ was carried out in

ref.[16], yielding at leading order the known 2D charged black hole heterotic string solutions of McGuigan, Nappi and Yost[20], and 4D dyonic solutions were defined by tensor product with the GPS theory. At about the same time, Lowe and Strominger wrote a paper[21] about 4D dyons which were defined by tensoring the GPS theory with an asymmetric orbifold of $SL(2, \mathbb{R})$. This asymmetric orbifold may be realised as one of the $SL(2, \mathbb{R})$ heterotic cosets described in ref.[16].

Instead of tensor products of these 2D theories, it is possible to obtain 4D dyon solutions which are not tensor products by gauging (for example) a $U(1) \times U(1)$ subgroup of $SL(2, \mathbb{R}) \times SU(2)$ embedded non-trivially such that the action of the $U(1)$'s was shared among the two parent groups. In this way I obtained in ref.[16] a 4D dyon with a throat with a non-trivial mixing of the angular and radial coordinates¹. It would have been difficult to construct such a non-trivial solution as an exact conformal field theory without the use of the heterotic coset technique.

9. Future directions

There are a *huge* number of avenues opened by allowing such freedom to gauge any subgroup of the WZW model's symmetries, obtaining consistency by adding heterotic fermions. I cannot list all of the things which occur to me here, but the general point is that it allows one to consider leaving important WZW symmetries untouched, which in turn leaves certain spacetime symmetries intact. In the simple GPS monopole example, or in the even simpler example of an uncharged 2-sphere in the last section, leaving the $SU(2)_L$ (or $SU(2)_R$) action untouched meant that a simple spacetime spherically symmetric system was obtained from an $SU(2)$ WZW.

This type of freedom will certainly lead to many more interesting heterotic string backgrounds. The search for more 4D cosmological heterotic string backgrounds

¹ From the form of the low energy solution, I conjectured that this was a dyonic, axionic analogue of the Taub-NUT solution of General Relativity. With Myers[22] this conjecture was later confirmed by explicitly generating the full solution by using first T and then S duality transformations on the GR solution and then extremising it. Kallosh, Kastor Ortin and Torma [23] also constructed this solution at around the same time.

seems a promising area to apply this technique to.

Of great interest is the problem of calculating the spectrum and partition function for these models. This will be of course a highly non-trivial combination of right $N = 2$ characters and general $N = 0$ characters. It is a hard problem to discover the heterotic modular invariant combinations algebraically of course (see e.g. ref.[24]), but there are promising signs that their consistent description as a WZW as described in this talk may provide some guidance. Work is in progress on this and related matters with Berglund, Kachru and Zaugg[25].

The problem of starting to map out the moduli space of $(0, 2)$ models can be attacked successfully by studying the marginal perturbations of these models. This is of course much easier when there exists a Lagrangian description of the type constructed here. Such marginal perturbations would help to find the geometrical interpretation of the neighbourhoods of these models, in the case of their use as string compactifications.

Marginal perturbations would also represent interesting geometrical freedom in some 4D solutions, where they correspond to such processes as widening the throat of some of the extremal solutions of the type mentioned in the last section, connecting onto the asymptotically flat 4D exterior solution[17].

There are of course many more questions which need to be answered about the moduli space of $(0, 2)$ conformal field theories. I hope that this construction may go some way to help to answer them.

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I would also like once again to thank Ed Witten for originally pointing out to me over a year ago that the GPS monopole theory might be described by a construction of the type described in this talk.

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